A Study of the Sum of Two Squares: Approaching the Problem from One or Two Angles

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Abstract:

This paper investigates the mathematical concept of the sum of two squares, exploring its properties and implications through one or two methodological approaches. We analyze classical results related to the representation of integers as sums of two squares, providing insights into conditions under which such representations are possible. By comparing different techniques—such as algebraic and geometric methods—we aim to highlight the strengths and limitations of each approach. Our findings contribute to a deeper understanding of the sum of two squares, illustrating its significance in number theory and its applications in various mathematical contexts.

Key Words: Sum of Two Squares, Factorization, Greatest Common Divisor, Primes

1. Introduction:

For almost two millennia, people have thought of writing a given natural number as the sum of two squares. A small number of square numbers that might be expressed as the sum of two squares were discovered by Pythagoras and his colleagues, known as Pythagoreans, during their investigation. 32 plus 42 equals 52, 52 plus 122 equals 132, 82 plus 152 equals 172, etc. These three integers are known as Pythagorean Triples because they constitute the side lengths of a certain right triangle in geometry. Three natural integers, a, b, and c, make up the Pythagorean Triple in general, so that a 2 + b 2 = c 2. We see that c 2 = c 2 + 02 when we take 02 = 0 as a square number. It follows that all square numbers may be expressed in a single fashion as the sum of two squares, where one of the squares is 0. However, certain natural numbers are not squares and may be expressed as the sum of two squares. One such number is 10, for instance, because 10 = 12 + 32. If and only if the prime factorisation of a natural number n includes even powers of primes of the type 4k + 3, we may be certain that n can be expressed as the sum of two squares. This is a necessary and sufficient condition for using the sum of two squares to represent a given natural number. According to this theory, the numbers that may be expressed as the sum of two squares are 0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32,...

Fermat and Albert Girard stated in the 17th century CE that every prime of the form 4k + 1 may be expressed as the sum of two squares in a special fashion. This is now known as Fermat's Christmas Theorem. This essay will provide some basic methods for expressing a given natural number as the sum of two squares in either two distinct ways or precisely one way.

2. Theorem 1:

If a, b, c, d are four numbers then

$$(ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2) \times (c^2 + d^2) \qquad (1)$$
$$(ad - bc)^2 + (ac + bd)^2 = (a^2 + b^2) \times (c^2 + d^2) \qquad (2)$$

Proof:

(ac - bd) 2 + (ad + bc) 2 = a 2 c 2 + b 2 d 2 + a 2 d 2 + b 2 c 2 = (a 2 + b 2) × (c 2 + d

2) proving (1). (2) can be proved similarly.

These two basic identities are called Diophantus Identities or Brahmagupta - Fibonacci Identities.

3. Theorem 2:

If N is a natural number and if N = (p 2 + q 2) × (r 2 + s 2) then there exists four numbers a, b, c, d such that N = a 2 + b 2 = c 2 + d 2 (3)

Proof:

If N = (p 2 + q 2) × (r 2 + s 2) then we choose a = pr - qs, b = ps + qr, c = pr + qs, d = ps - qr so that a 2 + b 2 = (pr - qs) 2 + (ps + qr) 2 = (p 2 + q 2) × (r 2 + s 2) = N and c 2 + d 2 = (pr + qs) 2 + (ps - qr) 2 = (p 2 + q 2) × (r 2 + s 2) = N proving (3). This completes the proof.

3.1 Corollary 1:

If either p = 0 or q = 0 in N = (p 2 + q 2) × (r 2 + s 2) then N can be written as sum of two squares in only one way.

Proof:

From (1), we get N = (p 2 + q 2) × (r 2 + s 2) = (pr - qs) 2 + (ps + qr) 2 (4) From (2), we get N = (p 2 + q 2) × (r 2 + s 2) = (ps - qr) 2 + (pr + qs) 2 (5) If p = 0, then from (4), we get N = (qr) 2 + (qs) 2 and from (5), we get N = (qr) 2 + (qs) 2 If q = 0, then from (4), we get N = (pr) 2 + (ps) 2 and from (5), we get N = (pr) 2 + (ps) 2 Hence in either case, we notice that N can be written as sum of two squares in only one way. This completes the proof.

3.2 Corollary 2:

If p = q in N = (p 2 + q 2) × (r 2 + s 2) then N can be written as sum of two squares in only one way.

Proof:

Assuming p = q, from (4) we obtain N = (p 2 + p 2) × (r 2 + s 2) = (pr - ps) 2 + (ps + pr) 2 and from (5) we obtain N = (p 2 + p 2) × (r 2 + s 2) = (ps - pr) 2 + (pr + ps) 2 = (pr - ps) 2 + (ps + pr) 2 . Thus N can be written as sum of two squares in only one way. This completes the proof.

4. Theorem 3:

If N is a natural number and if N = a 2 + b 2 = c 2 + d 2 then there exists four integers p, q, r, s such that N = (p 2 + q 2) × (r 2 + s 2) (6)

Proof:

Since N = a 2 + b 2 = c 2 + d 2 , considering mod 4 operation, we note that if a and b are even then both c and d are also even. Similarly if a and b are odd, then both c and d are also odd. If one of a or b is odd, then one of c or d must be odd. Let us assume that a be odd and c is odd such that a < c. Let (x, y) represent the greatest common divisor of x and y.

Let p = ((c + a)/2, (b + d)/2) and q = ((c - a)/2, (b - d)/2). Then there exists integers r and s such that c + a = 2pr, b + d = 2ps, c - a = 2qs, b - d = 2qr. From these equations, we have 4(p 2 + q 2) × (r 2 + s 2) = 4p 2 r 2 + 4p 2 s 2 + 4q 2 r 2 + 4q 2 s 2 = (c + a) 2 + (b + d) 2 + (b - d) 2 + (c - a) 2 = 2(c 2 + a 2) + 2(b 2 + d 2) = 2(a 2 + b 2) + 2(c 2 + d 2) = 2N + 2N = 4N.

Hence N = (p 2 + q 2) × (r 2 + s 2) proving (6). This completes the proof.

5. Theorem 4:

If N = p1 × p2 where both p1 and p2 are primes of the form 4k + 1, for some k, then N can be written as sum of two squares. In particular if p1 = p2 then we obtain a Pythagorean Triple and if p1 and p2 are distinct, then N can be expressed as sum of two squares in two different ways.

Proof:

By Fermat's Christmas Theorem we know that any prime of the form $4k + 1$ is expressible as sum of two squares in only one way. Hence if $p1$ and $p2$ are primes of the form $4k + 1$, for some k, then we can write $p1 = p^2 + q^2$ and $p2 = r^2 + s^2$ for some integers p, q, r, s. Then we can write $N = p1 \times p2 = (p^2 + q^2) \times (r^2 + s^2)$. By (3) of Theorem 2, there exists integers a, b, c, d such that $N = a^2 + b^2 = c^2 + d^2$, where $a = pr - qs$, $b = ps + qr$, $c = pr + qs$, $d = ps - qr$. Thus, N can be expressed as sum of two squares. From (4) and (5) we obtain $N = p1 \times p2 = (p^2 + q^2) \times (r^2 + s^2) = (pr - qs)^2 + (ps + qr)^2$ and $N = p1 \times p2 = (p^2 + q^2) \times (r^2 + s^2) = (ps - qr)^2 + (pr + qs)^2$. If $p1 = p2$ then let $p = r$ and $q = s$. Hence $N = (p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2$. Hence $(p^2 - q^2, 2pq, p^2 + q^2)$ forms a Pythagorean Triple. If $p1$ and $p2$ are distinct, then from $N = p1 \times p2 = (p^2 + q^2) \times (r^2 + s^2)$ the values $a = pr - qs$, $c = pr + qs$ will be distinct as well as $b = ps + qr$, $d = ps - qr$ will be distinct. Hence $N = a^2 + b^2 = c^2 + d^2$ would be two distinct ways of writing N. Thus, in this case, N can be expressed as sum of two squares in two different ways. This completes the proof.

6. Theorem 5:

If N can be expressed as sum of two squares in two different ways, then 2N is also expressible as sum of two squares in two different ways.

Proof:

Let $N = a^2 + b^2 = c^2 + d^2$ for some four numbers a, b, c, d. Now we consider the basic identity from algebra namely, $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$ and $(c + d)^2 + (c - d)^2 = 2(c^2 + d^2)$. Therefore, from $N = a^2 + b^2 = c^2 + d^2$ we obtain $2N = (a + b)^2 + (a - b)^2 = (c + d)^2 + (c - d)^2$. Thus, 2N is also expressible as sum of two squares. This completes the proof.

6.1 Corollary 3:

There exists infinitely many natural numbers which are sum of two squares in two different ways.

Proof:

From theorem 5, we know that if N is expressible as sum of two squares in two different ways, then 2N can also be done so. Similarly, by the same argument, we notice that 2(2N) = 4N, 2(4N) = 8N, 2(8N) = 16N, … are also expressible as sum of two squares in two different ways. Therefore, if N is expressible as sum of two squares in two different ways, then the numbers of the form 2kN for any natural number k, can also be expressed as sum of two squares in two different ways. This completes the proof.

Conclusion:

This paper's main goal is to provide simple techniques for determining if a given natural number may be expressed in one or two distinct ways as the sum of two squares. Numerous significant mathematicians have previously solved this issue, as stated in the Introduction. The generalised form of this issue has also been examined by Srinivasa Ramanujan, the mathematical genius from India. A new notion known as quadratic forms has emerged from these principles, which allow us to attempt to describe a given natural number N as a linear combination of squares with certain coefficients.

We have shown in this article, in Theorem 2, that if a natural number N is the product of two numbers, each of which is the sum of two squares, then their product is also a number that can be written as the sum of two squares in one or two distinct ways. We have listed the situations in which a number may be written as the sum of two squares in only one manner in the two corollaries that follow Theorem 2. The contrary half of the fact of Theorem 2 was demonstrated in Theorem 3. Specifically, one of the simplest ways to do so will be the proof of Theorem 3 presented in this work.

In Theorem 4, we specify the prime factorisation requirement for a given natural number N and demonstrate that, depending on the prime factors of N, the number N may always be expressed as the sum of two squares in one of two ways. Lastly, we proved that there exist an unlimited number of natural numbers that may be expressed as sums of two squares in two distinct ways in Theorem 5 and the following Corollary. Despite the fact that most of these findings have already been published, we believe that the new and simple approaches we have shown in this work will be useful, particularly for aspiring instructors and young researchers who want to delve further into this rich field of study.

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